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NONLINEAR MODEL OF THE INTERACTION OF CRYOAGENT FLOWS IN HEAT EXCHANGERS

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A nonlinear analytical model of a two-flow heat exchanger is developed, ensuring high accuracy and speed, and universal indexing of the initial temperatures and eliminating degeneration of the heat transfer.

In investigating the cooling and heatingh of cryogenic systems, and in describing various transient conditions of quasistatic type, there arises the problem of correctly determining the functional relations between the limiting temperature of a two-flow heat exchanger.

Traditional methods of solving steady-heat-transfer problems reduce, as a rule, to two schemes. According to the first, the initial heat-transfer equations are integrated under the assumption of constant properties and parameters of the heat-carrier interaction. The limiting temperatures obtained here allow the mean values of the corresponding "constants" to be refined, after which the desired temperature values are redefined, and so on (linear-averaged model) [1, 2]. In practice, this scheme is a multistep iterative process.

The second calculation scheme reduces to direct integration of the heat-transfer equations on a computer, automatically taking account of the change in the coefficients at each step. The initial temperatures are specified here at one end of the heat exchanger.

Recently, combined calculation schemes have also appeared [3]; in these schemes, some of the deficiencies of linear models are eliminated in a narrow parameter range close to nominal conditions, as calculated by numerical integration.

Analysis of these methods leads to the conclusion that calculation by the first is faster than calculation by the second and is more flexible from the viewpoint of the possibility of determining an arbitrary pair of limiting temperatures. This is often decisive in the choice of an algorithm for investigating systems with parallel and series combinations of heat exchangers. However, the artificial linearization of the distributed parameters in the first method may lead to fundamentally incorrect solutions. This is associated with the possible disregard of those regions of the heat exchanger where nonlinearity of the thermophysical properties of the flows may lead to intersection of the temperature profiles, which would mean that the intermediate temperature differences vanish. In reality (if the influence of hydrayulic losses, external heat sources, and heat conduction is neglected), such degeneracy cannot occur. This physically impermissible phenomenon may be eliminated by developing a nonlinear model of heat transfer. In addition, it must be emphasized that the nonlinear terms of the equations describing processes in heat exchangers for the low-temperature region ($T \sim 20$-4.5 K) amount to tens of percent with respect to the linear terms, which also points to a need to develop a nonlinear theory of heat transfer.

Consider an initial system of steady equations of a two-flow heat exchanger, omitting the terms due to hydraulic losses, external heat sources, and heat conduction. The calculation of each of these factors falls outside the scope of the present work and may be accomplished by classical methods. The influence of these factors on the heat transfer is assumed to be small, and is easily taken into account by perturbation theory [1, 2] for the solution given below, which is expediently interpreted as the nonlinear zero approximation. Thus, fixing the flow rates and mean pressure in each flow for steady conditions, it is found that

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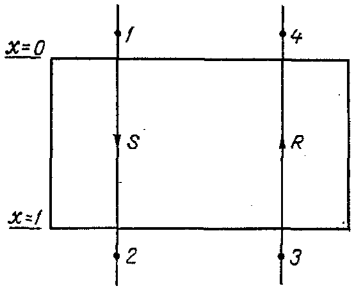

Fig. 1. Indexing of the boundary points and flows in a heat exchanger.

$$G_s \frac{dh_s}{dx} = G_R \frac{dh_R}{dx} = kF(T_R - T_s). \tag{1}$$

Analysis of the temperature–enthalpy curves of the coolant T(h) shows that, to achieve reasonable accuracy (≤1%), it is quite adequate to limit consideration to the third order of the polynomial approximation of the thermophysical properties, even close to the phase-transition point. The description of the cubic curve requires, as is known, the specification of four characteristic parameters. In choosing these parameters it must be taken into account that each call of the subprogram for finding the thermodynamic properties of the coolants allows both h(T) and $C_p(T)$ to be found (at fixed pressure) simultaneously, practically without losss of machine time. Thus, the required cubic function T(h) of each flow may be constructed from two points (the inlet and outlet), giving the desired four-parameter set. The use of this set is very expedient, both in computational terms and in terms of machine-time economy. As may readily be shown here

$$T(h) = T_m + (a - b + c)(h - h_m) + (b - 3c)\frac{(h-h_m)^2}{(h_n - h_m)} + 2c\frac{(h-h_m)^3}{(h_n-h_m)^2}, \tag{2}$$

where the subscripts m and n correspond to the two reference points of the heat exchanger (1, 2 in the forward flow or 3, 4 in the backward flow; Fig. 1); a characterizes the linear properties of the curve, b characterizes the parabolic deviations from linearity, and c characterizes the cubic deviations from parabolic form

$$a = \frac{T_n - T_m}{h_n - h_m};\quad b = \frac{1}{2}\left(\frac{1}{C_p^n} - \frac{1}{C_p^m}\right);\quad c = \frac{1}{2}\left(\frac{1}{C_p^n} + \frac{1}{C_p^m}\right) - a.$$

The subsequent integration of Eq. (1) requires adding the specific form of k(T) to the chosen approximation of the properties T(h). In the general case, k is determined not only by the temperatures but also by the pressures and flow rates of the coolants [4]. However, in the given static conditions (G = const), with small hydraulic losses, the only quantities which are changing markedly are the temperatures. Thus, the dependence of k on the flow rates and pressures is then taken into account parametrically rather than functionally. Estimates of the character of the dependences k(T(h)) for typical cryogenic-system heat exchangers show that, within the limits of reasonable accuracy (≲1%), the parabolic approximation of the corresponding curves is sufficient.

Taking account of this, after introducing the new variable $\xi = (h_S - h_1)/(h_2 - h_1) = (h_R - h_4)/(h_3 - h_4)$ and the approximation $(kf)^{-1} = A + (4B - 3A - C)\xi + 2(A + C - 2B)\xi^2$, integrating Eq. (1) leads to two algebraic equations

$$G_s(h_1 - h_2) = G_R(h_4 - h_3), \tag{3}$$

$$\int_0^1 \frac{d\xi\,[A + (4B - 3A - C)\xi + 2(A + C - 2B)\xi^2]}{Z + (U - V + W)\xi + (V - 3W)\xi^2 + 2W\xi^3} = 1, \tag{4}$$

where

$$Z = \frac{T_1 - T_4}{G_s(h_1 - h_2)};\quad U = \frac{T_4 - T_3}{G_R(h_4 - h_3)} - \frac{T_1 - T_2}{G_s(h_1 - h_2)};\quad V = \frac{1}{2}\left[\frac{1}{G_R}\left(\frac{1}{C_p^3} - \frac{1}{C_p^4}\right) - \frac{1}{G_s}\left(\frac{1}{C_p^2} - \frac{1}{C_p^1}\right)\right];$$

$$W = \frac{1}{2}\left[\frac{1}{G_R}\left(\frac{1}{C_p^3} + \frac{1}{C_p^4}\right) - \frac{1}{G_s}\left(\frac{1}{C_p^1} + \frac{1}{C_p^2}\right)\right] - U;\quad A = (kF)^{-1}|_{\xi=0};\quad B = (kF)^{-1}|_{\xi=\frac{1}{2}};\quad C = (kF)^{-1}|_{\xi=1}.$$

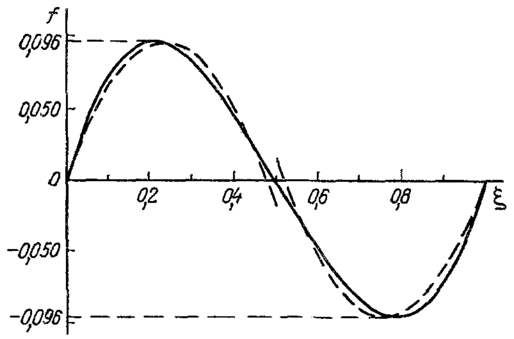

Fig. 2. Form of the function $f(\xi)$, characterizing the influence of cubic properties of the coolants (continuous curve) and the parabolic approximation (dashed curve).

Then $\xi = 0$ corresponds to $T_S = T_1$; $T_R = T_4$; $\xi = 1$ to $T_S = T_2$, $T_R = T_3$; and $\xi = 1/2$ to $T_S = T_S[(h_1 + h_2)/2]$, $T_R = T_R[(h_3 + h_4)/2]$. Within the framework of the cubic approximation, Eq. (2) readily yields the result

$$T\Big|_{\xi=\frac{1}{2}} = \frac{T_m + T_n}{2} - \left(\frac{h_n - h_m}{8}\right)\left(\frac{1}{C_p^n} - \frac{1}{C_p^m}\right).$$

In analyzing Eq. (4), note that the denominator of the integrand, which is cubic in ξ, is proportional to the current temperature difference $T_R - T_S$. Hence, it is clear that the possibility that the given cubic curve goes to zero more than once corresponds to the above-discussed situation of the "degeneracy" of heat transfer. The impermissibility of the appearance of zeros in the denominator of Eq. (4) imposes constraints on the minimum possible values of the underrecuperation $T_1 - T_4$ and $T_2 - T_3$, which automatically transfers the subsequent iterative process to the real range of desired temperatures.

It is also expedient to isolate the role of the parameters V and W, which are used, together with the inequalities $A \neq B \neq C$, for nonlinear correction of the functional relation between the limiting temperatures. In fact, when $V = W = 0$, and also $A = B = C$, Eqs. (3) and (4) are transformed after integration to the relations characteristic of linear theory [2].

The algorithm for solving Eqs. (3) and (4) reduces to the following: with two specified values of the boundary temperatures, the third point is selected from Eq. (4) and the fourth point required to make Eq. (4) specific is determined at each step from the enthalpy-balance Eq. (3). Thus, the given problem of searching for any pair of limiting temperatures in fact reduces to analytical integration of Eq. (4) and subsequent solution of a single transcendental equation with one unknown temperature.

Note that using a third-order polynomial in the determinant of the integrand in Eq. (4) leads to explicit overcomplication of the analytical form of the integral. However, discarding the cubic terms may entail considerable loss of accuracy. This dilemma may be resolved by noting that the cubic terms which are of interest here (the terms proportional to W) form the combination $Wf(\xi)$ in the denominator of Eq. (4), where $f(\xi) = 2\xi^3 - 3\xi^2 + \xi$. The form of this function (Fig. 2) clearly demonstrates the possibility of sufficiently accurate approximation by two parabolas, the coefficients of which are obtained by the least-squares method, fixing the values $f|_{\xi=0} = f|_{\xi=1} = 0$

$$f(\xi) = 2\xi^3 - 3\xi^2 + \xi = \begin{cases} -\dfrac{5}{3}\xi^2 + \dfrac{4}{5}\xi \text{ when } 0 \leqslant \xi < \dfrac{1}{2}, \\ \dfrac{5}{3}\xi^2 - \dfrac{38}{15}\xi + \dfrac{13}{15} \text{ when } \dfrac{1}{2} < \xi \leqslant 1. \end{cases}$$

The error of the final result due to this substitution is no more than 1%. With the same accuracy, the numerator of the integrand in Eq. (4) may be expediently replaced by a piecewise-linear function

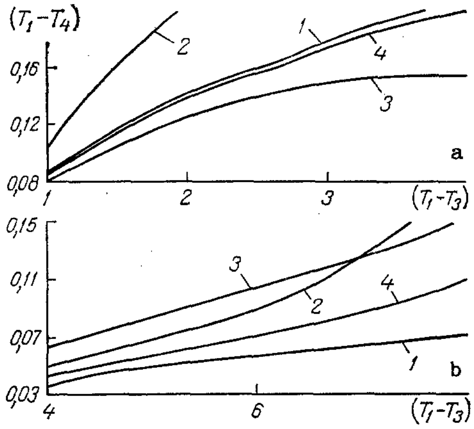

Fig. 3. Dependence of the underrecuperation $(T_1 - T_4)$ on the temperature difference $(T_1 - T_3)$ in a heat exchanger at different levels of the mean temperature $T_C = (T_1 + T_3)/2$: a) $T_C = 7$ K; b) 10 K.

$$A + 2(B-A)\xi \text{ when } 0 \leqslant \xi \leqslant \frac{1}{2},$$

$$(2B-C) - 2(C-B)\xi \text{ when } \frac{1}{2} \leqslant \xi \leqslant 1.$$

As a result of these manipulations, the integral in Eq. (4) may be written in the form of a combination

$$A\sigma_0(\alpha, \beta, \gamma) + (B-A)\sigma_1(\alpha, \beta, \gamma) + C\sigma_0(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}) + (B-C)\sigma_1(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}),$$

where

$$\alpha = \frac{V}{2} - \frac{5}{6}W, \quad \beta = U - V + \frac{4}{5}W, \quad \gamma = 2Z;$$

$$\tilde{\alpha} = \frac{V}{2} + \frac{5}{6}W, \quad \tilde{\beta} = -\left(U + V + \frac{4}{5}W\right), \quad \tilde{\gamma} = 2(Z + U).$$

Thus, the problem of integrating Eq. (4) reduces to calculating the simpler integrals σ_0 and σ_1, the analytical form of which is easily determined from tables

$$\sigma_n(\alpha, \beta, \gamma) = \int_0^1 \frac{\xi^n d\xi}{\alpha\xi^2 + \beta\xi + \gamma}. \tag{5}$$

The relations obtained above allow a program for computer calculation of the temperature operating conditions of heat exchangers to be developed, while satisfying two conflicting requirements: 1) high accuracy; 2) high speed. The results of using the nonlinear analytical model (curve 1), the linear model (curve 2), and the traditional model of direct numerical integration of Eq. (1), dividing the heat exchanger into M steps (curves 3 and 4), are compared in Fig. 3. A real heat exchanger of a cryogenic system, in which the flow rates of pressure of both coolant (helium) flows are fixed, is considered here. Figures 3a and 3b each correspond to a definite level of the mean temperature $T_C = (T_1 + T_3)/2$; the curves also show the dependence of the underrecuperation $T_1 - T_4$ as a function of the total temperature difference $T_1 - T_3$.

It is important to note that, with increase in the number of steps M to a few hundred, the direct-integration scheme (a particular kind of test) leads to practical superposition of the corresponding curves (M = 200, 500) into curve 1 (not shown in Fig. 3, for the sake of simplicity). In particular, this allows high accuracy of the given nonlinear model to be guaranteed, and reliably demonstrates how much this model differs from step schemes (with a small number of divisions M) and linear methods (curve 2), which clearly do not meet current requirements.

To compare the speed of calculations by numerical and analytical schemes, note that calculation algorithms require, as a rule, the specification of the input temperatures (i.e., for a heat exchanger, points 1 and 3; Fig. 1), and consequently the step-integration scheme (with 1, 4 or 2, 3 as the initial points) entail 15-20 repetitions of this procedure (depending on the accuracy desired) in order to select one of the points (1 or 3). The universality of the nonlinear model with respect to the indexing of the initial points, by contrast, does not require a double iterative process, and the time for the corresponding calculation is equivalent to a process which division into no more than 3 steps. Thus, the nonlinear analytical model permits an increase in speed by 2-3 orders of magnitude in comparison with the step-integration scheme, thereby guaranteeing an accurate result, within the limits of correctness of the information on the initial parameters.

The nonlinear model of heat transfer here developed is sufficiently universal, and covers various types of heat exchangers. In particular, the problem of calculating single-flow heat exchangers in which the backward-flow function corresponds to the surrounding medium, i.e., $T_4 = T_3$, reduces to the integral in Eq. (4); in this case all the terms associated with points 3 and 4 in Eq. (4) must be omitted, formally letting $G_R \to \infty$. Heat exchangers of immersional type (nitrogen, helium baths) are described analogously, under the condition of sufficiently large heat-transfer coefficients from the boiling liquid.

NOTATION

G, coolant flow rate, kg/sec; T, current temperature, K; h, specific enthalpy, J/kg; C_p, isobaric specific heat, J/kg·K; k, local heat-transfer coefficient, W/m^2·K; F, heat-transfer surface, m^2; x, dimensionless coordinate varying from 0 at one end of the heat exchanger (points 1 and 4) to 1 at the other end (points 2 and 3; Fig. 1): Indices: S, forward flow (input, point 1; output, point 2); R, backward flow (input, point 3; output, point 4).

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